GIRRAWEEN HIGH SCHOOL

HALF-YEARLY EXAMINATION

YEAR 12

2013

MATHEMATICS

Time allowed – Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt all questions.
- Circle the best response for the questions in Part A.
- Start each question in Part B on a new page.
- All necessary working must be shown.
- Marks may be deducted for careless or badly arranged work.

PART A (10 marks)

Question 1

Evaluate $\sum_{n=3}^{6}(2n+1)$

(a) 7 (b) 11 (c) 13 (d) 40

Question 2

The limiting sum of the series $1+\frac{1}{2}+\frac{1}{4}+........$is:

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

Question 3

A person buys 3 tickets in a lottery in which 20 tickets are sold. If there are two prizes, find the probability that he wins at least one prize.

(a) $\frac{1}{20}$ (b) $\frac{27}{95}$ (c) $\frac{3}{190}$ (d) $\frac{17}{20}$

Question 4

The perpendicular distance from the point $(-1,\ 3)$and the line $4x+5y+6=0$ is:

(a) $\frac{25}{\sqrt{41}}$ (b) $\frac{17}{9}$ (c) $\frac{17}{\sqrt{41}}$ (d) $\frac{25}{9}$

Question 5

The equation of the line that has an angle of inclination of 45° to the $x-axis$ and a y intercept of 2 is:

(a) $x+y+2=0$ (b) $x-y+2=0$ (c) $2x+y+2=0$ (d) $2x-y+2=0$

Question 6

The second derivative of $y=e^x$ is:

(a) e^x (b) xe^x (c) $2e^x$ (d) $2xe^x$

Question 7

Evaluate $\lim_{t\to 4}\left(t^2-2t+1\right)$

(a) 7 (b) 9 (c) 10 (d) 16

Question 8

If $x^2 - 2x + 5 = 0$ then $\alpha + \beta$ is:

(a) 2 (b) -2 (c) 5 (d) -5

Question 9

If a parabola has an equation $x^2 = 8y$, it has a focal length of:

(a) -2 (b) 2 (c) -8 (d) 8

Question 10

Evaluate $\int_0^2 (2x-1)^3 dx$

(a) 8 (b) $\frac{81}{8}$ (c) 10 (d) 16

PART B

Question 11 (17 marks)

(a) Differentiate the following. Simplify the answer if necessary.

(i) $y = 4x^3 + \sqrt{x}$ **2**

(ii) $y = e^{5x-2}$ **1**

(iii) $y = \frac{2x+1}{e^x}$ **3**

(iv) $y = (x^2 - 3)^2$ **3**

(v) $y = xe^{x^2}$ **3**

(vi) $y = \frac{1}{2}(e^x - e^{-x})$ **3**

(b) Find the equation of the parabola with focus $(2, 0)$ and directrix $x = -2$. **2**

Question 12 (13 marks)

(a) $A(4,14)$, $B(6,10)$, $C(14,14)$ are 3 vertices of a parallelogram *ABCD*.

(i) Find the co-ordinates of *D*. **1**

(ii) Find the midpoint *M* of *AC*. **2**

(iii) Write down the equation of the circle with *AC* as the diameter. **3**

(iv) Show that *D* lies on the circle. **1**

(b) A biased coin is tossed three times. If the probability of getting a head is $\frac{2}{5}$, find the probability of getting:

(i) Three heads **2**

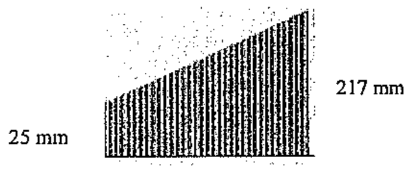
(ii) Two heads then a tail. **2**

(iii) Two heads and a tail.

(iv) At least one head. **2**

Question 13 (11 marks)

(a) A logo is made with vertical lines equally spaced as shown. The shortest line is 25 mm, the longest is 217 mm and the sum of the lengths of all the lines is 5929 mm.



(i) How many lines are in the logo? **2**

(ii) Find the difference in length between adjacent lines. **2**

(b) Use the Trapezoidal rule to find an approximation for $\int_1^2 \frac{1}{x}dx$ using 4 sub-intervals. **4**

(c) Find the volume enclosed by the surface generated when the curve $x^2 + y^2 = 16$ is rotated about the $x-axis$ between $x = 0$ and $x = 4$. **3**

Question 14 (16 marks)

(a) Find the following integrals.

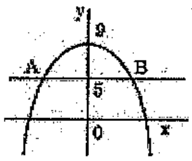
(i) $\int 4x^3 - 6x + 7dx$ **1**

(ii) $\int \frac{2x^2 - 8x}{4x} dx$ **2**

(iii) $\int e^{5x} dx$ **1**

(iv) $\int_0^2 xe^{x^2} dx$ **3**

(b) The line $y = 5$ meets the parabola $y = 9 - x^2$ at A and B as shown in the diagram.



Find the

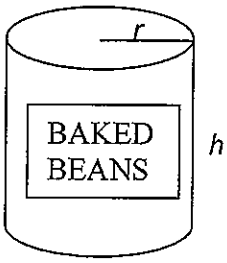
(i) Co-ordinates of A and B. **3**

(ii) Area bounded by the line $y = 5$ and the parabola. **3**

(iii) Volume generated by the area in (ii) when rotated about the $y - axis$. **3**

Question 15 (16 marks)

(a) A can of baked beans is in the shape of a closed cylinder with height h cm and radius r cm, as shown in the diagram.



(i) The volume of the can is 500 cm^3. Find an expression for h in terms of r. **2**

(ii) Show that the surface area, S cm^2, of the can is given by

$$S = 2\pi r^2 + \frac{1000}{r}$$ **3**

(iii) Hence, if the area of metal used to make the can is to be minimized, find the radius of the can. **4**

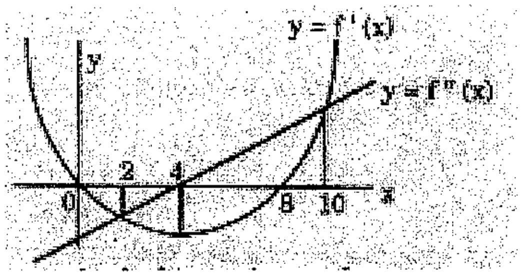
(b) A surveyor makes a straight line traverse across the bend of a creek and measures perpendicular offsets every two metres as shown. By application of Simpson's Rule, find the approximate area bounded by the traverse line and the creek. **3**

Intervals	0	2	4	6	8	10	12	14	16	18	20
Width of creek	0	7.12	8.92	10.62	10.85	9.62	8.76	6.30	3.21	2.62	0

(c) If $y=(1+3x)e^{2x}$, show that $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$. **4**

Question 16 (17 marks)

(a) The graphs of the first and second derivatives of the curve $y=f(x)$ are shown in the diagram.



(i) Write down the x co-ordinates of the stationary points of $y=f(x)$ and determine their nature. **3**

(ii) Find the x co-ordinates of any point(s) of inflexion of $y=f(x)$. **2**

(iii) Write down the domain for which $y=f(x)$ is monotonic decreasing and concave down. **2**

(b) Consider the curve $y=xe^{-x}$

(i) Find any stationary points and determine their nature. **4**

(ii) Find any points of inflexion. **3**

(iii) Sketch the curve in the domain $x\geq 0$. **3**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12 2 UNIT HALF-YEARLY

PART A

1. D
2. C
3. B
4. C
5. B
6. A
7. B
8. A
9. B
10. C

PART B

Q11.

a i) $y = 4x^3 + \sqrt{x}$

$y' = 12x^2 + \frac{1}{2}x^{-\frac{1}{2}}$

$= 12x^2 + \frac{1}{2\sqrt{x}}$

ii) $y = e^{5x-2}$

$y' = 5e^{5x-2}$

iii) $y = \frac{2x+1}{e^x}$

$u = 2x+1$, $u' = 2$, $v = e^x$, $v' = e^x$

$y' = \frac{e^x(2) - (2x+1)e^x}{(e^x)^2}$

$= \frac{e^x(2 - 2x - 1)}{e^{2x}}$

$= \frac{e^x(1-2x)}{e^{2x}} = \frac{1-2x}{e^x}$

iv) $y = (x^2-3)^2$

$u = x^2 - 3$, $\frac{du}{dx} = 2x$

$y = u^2$, $\frac{dy}{du} = 2u$

$y' = 2x(2u)$

$= 2x \times 2(x^2-3)$

$= 4x(x^2-3)$

$= 4x^3 - 12x$

v) $y = xe^{x^2}$

$u = x$, $u' = 1$, $v = e^{x^2}$, $v' = 2xe^{x^2}$

$y' = e^{x^2}(1) + x(2xe^{x^2})$

$= e^{x^2} + 2x^2e^{x^2}$

$= e^{x^2}(1 + 2x^2)$

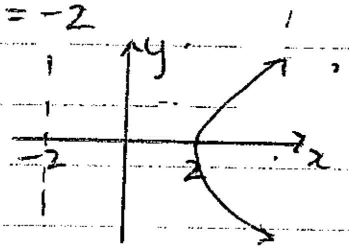
vi) $y = \frac{1}{2}(e^x - e^{-x})$

$y' = \frac{e^x}{2} + \frac{e^{-x}}{2}$

$= \frac{e^x + e^{-x}}{2}$

b) $S(2,0)$ $x = -2$

$y^2 = 8x$



Q12.

ai) $D(12, 18)$

ii) $A(4, 14)$ $C(14, 14)$

$$M_{AC} = \left(\frac{4+14}{2}, \frac{14+14}{2}\right)$$

$$= (9, 14)$$

iii) AC is diameter

$M_{AC} = (9, 14)$

$\therefore M_{AC}$ is the centre of the circle.

$$d_{AM} = \sqrt{(4-9)^2 + (14-14)^2}$$

$$= \sqrt{25} = 5$$

$\therefore$ radius is 5.

$\therefore$ The equation of the circle is $(x-9)^2 + (y-14)^2 = 25$

iv) $D(12, 18)$

$$LHS = (x-9)^2 + (y-14)^2$$

$$= (12-9)^2 + (18-14)^2$$

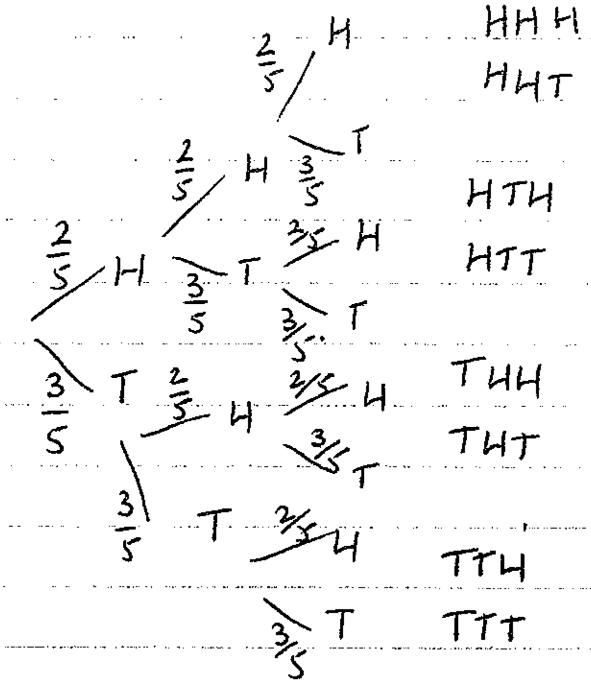
$$= 9 + 16$$

$$= 25$$

$$= RHS$$

$\therefore$ D lies on the circle.

b) $P(H) = \frac{2}{5}$ $P(T) = \frac{3}{5}$.



i) $P(HHH) = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$

ii) $P(HHT) = \left(\frac{2}{5}\right)^2 \times \frac{3}{5} = \frac{12}{125}$

iii) $P(2H, T) = HHT + HTH + THH$

$$= \left(\frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5} \times \frac{2}{5}\right) + \left(\frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}\right)$$

$$= \frac{12}{125} + \frac{12}{125} + \frac{12}{125} = \frac{36}{125}$$

iv) $P(\text{at least } 1H) = 1 - P(TTT)$

$$= 1 - \left(\frac{3}{5}\right)^3$$

$$= 1 - \frac{27}{125} = \frac{98}{125}.$$

Q13.

a) $a = 25 \quad l = 217 \quad S_n = 5929$

i) $S_n = \frac{n}{2}(a + l)$

$5929 = \frac{n}{2}(25 + 217)$

$11858 = 242n$

$\therefore n = 49$

$\therefore$ There are 49 lines in the logo.

ii) $T_1 = 25 \qquad T_{49} = 217$

$a = 25 \qquad 217 = a + 48d$

$\therefore 217 = 25 + 48d$

$48d = 192$

$\therefore d = 4.$

b) $y = (1+3x)e^{2x}$

$= e^{2x} + 3x e^{2x}$

$y' = 2e^{2x} + [e^{2x}(3) + 3x(2e^{2x})]$ $\quad u = 1+3x$

$= 2e^{2x} + 3e^{2x} + 6xe^{2x}$ $\quad u' = 3$

$= 5e^{2x} + 6xe^{2x}$ $\quad v = e^{2x}$

$v' = 2e^{2x}$

$y'' = 10e^{2x} + [e^{2x}(6) + 6x(2e^{2x})]$

$= 10e^{2x} + 6e^{2x} + 12xe^{2x}$

$= 16e^{2x} + 12xe^{2x}$

$LHS = \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y$

$= (16e^{2x} + 12xe^{2x})$

$- 4(5e^{2x} + 6xe^{2x})$

$+ 4(1+3x)e^{2x}$

$= 16e^{2x} + 12xe^{2x} - 20e^{2x} - 24xe^{2x}$

$+ 4e^{2x} + 12xe^{2x}$

$= 0$

$= RHS.$

b) $\int_1^2 \frac{1}{x}\,dx \qquad h = \frac{2-1}{5} = \frac{1}{5}.$

x	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
	0	5	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	1

$\int_1^2 \frac{1}{x}\,dx \doteqdot \frac{0.2}{2}\left[0 + 1 + 2\left(5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4}\right)\right]$

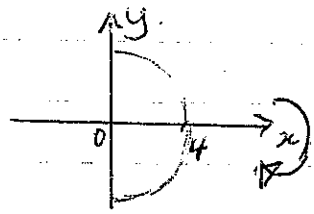
$= \frac{1}{10}\left[1 + 10 + \frac{10}{2} + \frac{10}{3} + \frac{10}{4}\right]$

$= \frac{1}{10} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$= 2.183$ (3dp)

c) $x^2 + y^2 = 16$

$y^2 = 16 - x^2$



$V = \pi \int_0^4 16 - x^2\,dx$

$= \pi\left[16x - \frac{x^3}{3}\right]_0^4$

$= \pi\left(16(4) - \frac{4^3}{3}\right)$

$= \frac{128}{3}\pi$ units^3

Q14.

ai) $\int 4x^3 - 6x + 7\, dx$

$= x^4 - 3x^2 + 7x + C$

ii) $\int \frac{2x^2 - 8x}{4x}\, dx$

$= \int \frac{x}{2} - 2\, dx$

$= \frac{x^2}{4} - 2x + C$

iii) $\int e^{5x}\, dx = \frac{1}{5} e^{5x} + C$

iv) $\int_0^2 x e^{x^2}\, dx$

$= \frac{1}{2} \int_0^2 2x e^{x^2}\, dx$

$= \frac{1}{2} \left[e^{x^2} \right]_0^2$

$= \frac{1}{2} (e^4 - 1)$

bi) $y = 5 \quad y = 9 - x^2$

$9 - x^2 = 5$

$x^2 = 4$

$\therefore x = \pm 2$

When $x = 2 \qquad x = -2$

$y = 5 \qquad y = 5$

$\therefore A(-2, 5) \quad B(2, 5)$

ii) $\int_{-2}^{2} (9 - x^2) - 5\, dx$

$= 2 \int_0^2 9 - x^2 - 5\, dx$

$= 2 \int_0^2 4 - x^2\, dx$

$= 2 \left[4x - \frac{x^3}{3} \right]_0^2$

$= 2 \left[\left(8 - \frac{8}{3}\right) - 0 \right]$

$= 2 \left(\frac{16}{3} \right) = \frac{32}{3}$ units^2

iii) $V = \pi \int_5^9 9 - y\, dy$ $\qquad y = 9 - x$ $\quad x^2 = 9 -$

$= \pi \left[9y - \frac{y^2}{2} \right]_5^9$

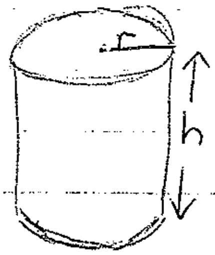
$= \pi \left[\left(81 - \frac{81}{2}\right) - \left(45 - \frac{25}{2}\right) \right]$

$= \pi \left(\frac{81}{2} - \frac{65}{2} \right)$

$= 8\pi$ units^3.

Q15.

a) i) $V = 500\text{cm}^3$



$V = \pi r^2 h$.

$500 = \pi r^2 h$.

$\therefore h = \frac{500}{\pi r^2}$.

ii) $SA = 2\pi r^2 + 2\pi r h$.

$= 2\pi r^2 + 2\pi r \left(\frac{500}{\pi r^2}\right)$.

$= 2\pi r^2 + \frac{1000\pi r}{\pi r^2}$.

$= 2\pi r^2 + \frac{1000}{r}$.

iii) $\frac{dA}{dr} = 4\pi r - \frac{1000}{r^2}$.

$\frac{dA}{dr} = 0 \Rightarrow 0 = 4\pi r - \frac{1000}{r^2}$.

$\frac{1000}{r^2} = 4\pi r$

$4\pi r^3 = 1000$.

$r^3 = \frac{1000}{4\pi}$.

$\therefore r = \sqrt{\frac{1000}{4\pi}}$.

$= \frac{10}{\sqrt[3]{4\pi}}$.

$\frac{d^2A}{dr^2} = 4\pi + \frac{2000}{r^3}$.

When $r = \frac{10}{\sqrt[3]{4\pi}}$

$\frac{d^2A}{dr^2} > 0 \quad \therefore$ min. value

$\therefore$ The radius is $r = \frac{10}{\sqrt[3]{4\pi}}$.

b) $h = \frac{20-0}{10} = 2$.

$A \doteqdot \frac{2}{3}\left[4(7.12 + 10.62 + 9.62 + 6.30 + 2.62) + 2(8.92 + 10.85 + 8.76 + 3.21)\right]$.

$= \frac{2}{3}\left(4(36.28) + 2(31.74)\right)$

$= \frac{2}{3}(145.12 + 63.48)$

$= 139.1\ \text{m}^2$ (1dp).

c) $y = (1+3x)e^{2x}$

$= e^{2x} + 3xe^{2x}$

$u = 1+3x$
$u' = 3$.
$v = e^{2x}$
$v' = 2e^{2x}$

$y' = 2e^{2x} + \left[e^{2x}(3) + 3x(2e^{2x})\right]$

$= 2e^{2x} + 3e^{2x} + 6xe^{2x}$

$= 5e^{2x} + 6xe^{2x}$

Q15c) cont.

$y'' = 10e^{2x} + \left[e^{2x}(6) + 6x(2e^{2x})\right]$

$= 10e^{2x} + 6e^{2x} + 12xe^{2x}$

$= 16e^{2x} + 12xe^{2x}$

$LHS = \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y$

$= (16e^{2x} + 12xe^{2x}) - 4(5e^{2x} + 6xe^{2x})$
$\quad + 4(1+3x)e^{2x}$

$= 16e^{2x} + 12xe^{2x} - 20e^{2x} - 24xe^{2x}$
$\quad + 4e^{2x} + 12xe^{2x}$

$= 0$.

$= RHS$.

Q16.

a i) stationary pts $f'(x) = 0$
when $x = 0$, $x = 8$
at $x = 0$ $f''(x) < 0$
$\therefore$ maximum turning pt.

at $x = 8$ $f''(x) > 0$
$\therefore$ Minimum turning pt.

ii) When $f''(x) = 0$ and sign changes.

$\therefore$ point of inflexion at $x = 4$.

iii) $f'(x) < 0$ $f''(x) < 0$

$\therefore D: 0 < x < 4$

b) $y = xe^{-x}$

i) $y' = (e^{-x})(1) + x(-e^{-x})$

$= e^{-x} - xe^{-x}$

| $u = x$
| $u' = 1$
| $v = e^{-x}$
| $v' = -e^{-x}$

$y' = 0$

$e^{-x} - xe^{-x} = 0$

$e^{-x}(1 - x) = 0$

$\therefore x = 1$

When $x = 1$ $y = (1)e^{-1}$
$= \frac{1}{e}$

$\therefore$ stationary pt $(1, \frac{1}{e})$

$y'' = -e^{-x} - (e^{-x} - xe^{-x})$
$= -e^{-x} - e^{-x} + xe^{-x}$
$= xe^{-x} - 2e^{-x}$

When $x = 1$ $y'' < 0$
$\therefore (1, \frac{1}{e})$ is a maximum turning pt.

ii) $y'' = 0$

$xe^{-x} - 2e^{-x} = 0$

$e^{-x}(x - 2) = 0$

$\therefore x = 2$

when $x = 2$ $y = 2e^{-2} = \frac{2}{e^2}$

Test

x	1	2	3
y''	$-\frac{1}{e^2}$	0	$\frac{1}{e^3}$.

$\therefore (2, \frac{2}{e^2})$ is a point of inflexion.

iii)